$\Phi_{k+1,k}$, $\Phi'_{k+1,k}$, $F_{k+1,k}$ $F'_{k+1,k}$, $G_{k+1,k}$	are the transition matrices;
$P_{k+1\|k+1}$, $P_{k+1\|k}$, Q_k, R_{k+1}	are the covariant matrices;
K_{k+1}	is the weighing matrix;
H_{k+1}	is the measurement matrix;
λ	is the thermal conductivity;
α	is the thermal diffusivity;
I	is the unit matrix;
τ	is the time;
h	is the grid interval.

LITERATURE CITED

1. R. Kalman and R. Byusi, "New results in linear filtering and prediction theory," Proc. Am. Soc. Mech. Eng., Ser. D, Tech. Mech., 83, No. 1 (1961).
2. D. F. Simbirskii, Temperature Diagnostics of Motors [in Russian], Tekhnika (1976).
3. Yu. M. Matsevityi, V. A. Malyarenko, and A. V. Multanovskii, "Problems of identification in nonlinear heat conduction problems," Teplomassoobmen-V, Vol. 9, Minsk (1976).

OPTIMAL CONTROL OF THE PROCESS OF HEAT TRANSMISSION BETWEEN BODIES IN CONTACT

V. S. Kolesov

UDC 536.24.02

Problems of optimal fast-response control of the process of heat transmission between bodies in contact are considered under constraints on the thermoelastic stresses. Analytic expressions are obtained for the control function – the thermal contact resistance.

The process of heat transmission between bodies in contact is characterized by the presence of a thermal resistance in the contact. It is due to the natural roughness of the surfaces in contact and can result in a substantial redistribution of the temperature fields in the materials making contact [1]. The influence of the thermal resistance in the contact on the heat transmission process is twofold: On the one hand, it diminishes the heat flux and therefore results in an increase in the lifetime of the process, and on the other hand, it reduces the temperature drop in the bodies making contact, i.e., results in a diminution in the thermal stress level therein. The dual nature of the influence of the thermal contact resistance on the heat-transmission process permits formulation of an optimal control problem: Find that control (the time dependence of the thermal resistance) which will result in a minimum time in the attainment of the desired result (the target function) and the temperature stresses will hence not exceed a certain quantity governing the strength of the material. The target function can be quite different. For example, the deviation of the mean body temperature from a previously assigned value will not exceed a certain quantity; a definite temperature level will be achieved at a fixed point, etc. Its selection is dictated by specific circumstances. Such problems originate in the design and designation of the exploitational modes of thermal power plants.

A significant number of investigations have been devoted to methods of solving problems on the optimal control of heating solids. The approach developed in [2, 3], whose crux is that compliance with the equality under conditions constraining the thermal stresses is considered equivalent to the condition of realizing an optimal thermal mode, is used below.

1. Let us assume the process of heat transmission between two halfspaces. This problem can be useful if it is necessary to check the process only at times close to the initial time, or when the items making contact are sufficiently massive.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 35, No. 4, pp. 718-723, October, 1978. Original article submitted October 3, 1977.

0022-0841/78/3504-1240$07.50

Let the halfspace $x > 0$ have the temperature T_0 and at the time $t = 0$ be in contact with the halfspace, $x < 0$ which has a zero temperature. Find that heat-transmission mode for which the value of the temperature at some point $x = l$ will reach the value $T^* (T_0 > T^* > T_0/2)$ in a minimum time and the thermoelastic stresses will hence not exceed an admissible quantity σ_0.

The heat-conduction problem is formulated as follows: Find the solution of the equations

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\varkappa} \frac{\partial T_1}{\partial t}, \quad x > 0; \quad \frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\varkappa} \frac{\partial T_2}{\partial t}, \quad x < 0, \tag{1.1}$$

satisfying the boundary conditions

$$T_1 = T_0, \ T_2 = 0 \ \text{for} \quad t = 0; \quad \frac{\partial T_1}{\partial x} = \frac{\partial T_2}{\partial x}, \\ \lambda \frac{\partial T_1}{\partial x} = \frac{1}{R}(T_1 - T_2) \ \text{for} \quad x = 0. \tag{1.2}$$

Let us first note one physically obvious equality (its formal proof is elementary), which will be useful later:

$$T_1(x, t) + T_2(-x, t) = T_0. \tag{1.3}$$

Furthermore, according to [4], the thermoelastic stresses in a halfspace with a uniform temperature distribution have the form

$$\sigma_{xx} = 0, \ \sigma_{zz} = \sigma_{yy} = -\frac{\alpha E T}{1-\nu}. \tag{1.4}$$

Therefore, the maximum value of the tensile stresses is achieved in the right halfspace and equals

$$\sigma_{zz}^{\max} = -\frac{\alpha E (T_1 - T_0)}{1 - \nu}\bigg|_{x=+0}. \tag{1.5}$$

Examination of the problem of optimal control of the heat-transmission process is meaningful only if the corresponding thermoelastic stresses exceed an allowable quantity under ideal thermal contact between the touching bodies. As follows from (1.2) and (1.3), a constant temperature $T_1 = T_2 = T_0/2$ and $\sigma_{zz}^{\max} = \alpha E T_0/2(1 - \nu)$ are established on the boundary under ideal thermal contact. Therefore, for $\alpha E T_0/2(1 - \nu) \le \sigma_0$ ideal heat exchange is the optimal heat-transmission mode. In this case the heat-conduction problem is formulated thus:

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\varkappa} \frac{\partial T_1}{\partial t}, \quad T_1|_{t=0} = T_0, \ T_1|_{x=0} = \frac{T_0}{2}. \tag{1.6}$$

The solution of this problem is known [5]:

$$T_1 = \frac{T_0}{2}\left\{1 + \Phi\left(\frac{x}{2\sqrt{\varkappa t}}\right)\right\},$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi.$$

The time needed to reach the temperature T^* at the point $x = l$ is determined from the equation

$$T^* = \frac{T_0}{2}\left\{1 + \Phi\left(\frac{l}{2\sqrt{\varkappa t}}\right)\right\}.$$

Let us consider the situation when $\alpha E T_0/2(1 - \nu) > \sigma_0$. Following [2, 3], let us assume that the optimal heat-transmission mode is realized under the condition of the equality $\sigma_{zz}^{\max} = \sigma_0$ is always satisfied at some point of the halfspace, where this point can be variable in the general case. However, taking account of the monotonity of the temperature change for the problem under investigation, it follows from (1.4) that the maximum tensile stress is always attained at the point $x = +0$. Therefore, under the assumption made is characterized by the condition

$$T_1|_{x=0} = T_0 - \frac{\sigma_0 (1 - \nu)}{\alpha E} = M$$

and the problem of determining the optimal mode of the heat-transmission process has the form

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\varkappa} \frac{\partial T_1}{\partial t}, \quad T_1|_{t=0} = T_0, \ T_1|_{x=0} = T_0 - \frac{\sigma_0 (1 - \nu)}{\alpha E}. \tag{1.7}$$

Let us prove that this thermal mode is actually optimal. The proof is based on the following lemma: Let functions $u_i(x, t)$ be solutions of the problems

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} \ (x, t > 0;\ i = 1,2),\ u_i|_{t=0} = u_0,\ u_i|_{x=0} = \varphi_i(t).$$

If $\varphi_1(t) \geq \varphi_2(t)$, then $u_1(x, t) \geq u_2(x, t)$.

Now, let $R(t)$ be some control and $T_1(x, t, R)$ be the thermal mode it determines. Let $\varphi(t, R) = T_1(0, t, R)$. If it turns out that for $R_1(t)$ and $R_2(t)$ the corresponding functions $\varphi_1(t, R_1)$ and $\varphi_2(t, R_2)$ are related by the inequality $\varphi_1 \geq \varphi_2$ then by virtue of the lemma $T_1(x, t, R_1) \geq T_1(x, t, R_2)$. Therefore, the control will be optimal if the function $T_1(x, t)$ is the greatest possible for $x = 0$. But the largest possible value is M. Therefore, the thermal mode it determines is optimal.

The solution of problem (1.7) is analogous to (1.6):

$$T_1 = T_0 - \frac{\sigma_0(1-\nu)}{\alpha E}\Phi^*\left(\frac{x}{2\sqrt{\varkappa t}}\right),\ T_2 = \frac{\sigma_0(1-\nu)}{\alpha E}\Phi^*\left(\frac{\varkappa}{2\sqrt{\varkappa t}}\right),$$

where

$$\Phi^*(x) = 1 - \Phi(x).$$

We find control function $R(t)$ realizing the optimal control from boundary condition (1.2)

$$R(t) = \frac{\alpha E T_0 - 2\sigma_0(1-\nu)}{\lambda\sigma(1-\nu)} 2\sqrt{\varkappa t}.$$

The time t^* to achieve the desired result is the solution of the equation

$$T_0 - T^* = \frac{\sigma_0(1-\nu)}{\alpha E}\Phi^*\left(\frac{l}{2\sqrt{\varkappa t}}\right).$$

2. Let us examine the more physical situation when two plates of identical thickness $2l$ are in contact, where one has the initial temperature T_0 while the temperature of the other is zero and the surfaces not in contact are heat insulated. Let us find the optimal heat-transmission mode between the plates by considering the contact thermal resistance $R(t)$ the control function, and the attainment of a certain temperature T^* on the heat-insulated surface of the "hot" plate in a minimum time as the target function, where $T_0/2 < T^* < T_0$.

The temperature distribution in the plates is described by the solution of the problem

$$\begin{gathered}\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\varkappa}\frac{\partial T_1}{\partial t},\ 0 < x \leqslant 2l;\ \frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\varkappa}\frac{\partial T_2}{\partial t},\ -2l \leqslant x < 0;\\ T_1 = T_0,\ T_2 = 0 \text{ for } t = 0;\ \frac{\partial T_1}{\partial x} = \frac{\partial T_2}{\partial x} \text{ and}\\ \lambda\frac{\partial \dot{T}_1}{\partial x} = \frac{1}{R}\left(T_1 - T_2\right) \text{ for } x = 0;\ \left.\frac{\partial T_1}{\partial x}\right|_{x=2l} = \left.\frac{\partial T_2}{\partial x}\right|_{x=-2l} = 0.\end{gathered} \tag{2.1}$$

Equality (1.3) is evidently also valid for this case.

Let us analyze the nature of the thermal stress state of plates with an ideal thermal contact in order to determine the condition under which the heat-transmission problem must be controlled by means of the thermal resistance. The mathematical formulation of the heat-conduction problem for $R = 0$ is analogous to (1.6) in conformity with (1.3), except the condition of no heat flux on the plate surfaces not in contact should be added. It is shown in [4] that the stresses in a free plate having the temperature T_0 at the initial instant and then changing its magnitude to S_0 instantaneously on one surface, will reach the maximum value on the boundary $x = 0$ at the time $t = +0$, where

$$\sigma_{zz}^{\max} = -\alpha E(S_0 - T_0)/(1-\nu),$$

i.e., for the problem under consideration

$$\sigma_{zz}^{\max} = \alpha E T_0/2(1-\nu).$$

TABLE 1. Values Roots of (2, 4)

m	β_m	m	β_m	m	β_m	m	β_m
1	1,30310	5	8,50113	9	14,8517	13	21,1567
2	3,70722	6	10,1184	10	16,4352	14	22,7339
3	5,28103	7	11,6898	11	18,0062	15	24,3049
4	6,93942	8	13,2803	12	19,5858	16	25,8803

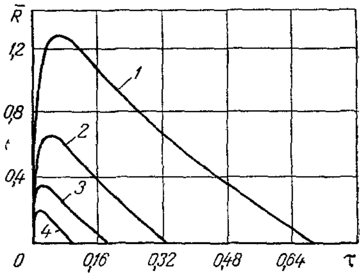

Fig. 1. Graph of the optimal control for different values of the parameter γ: 1) 0.2; 2) 0.3; 0.4; 3) 0.4; 4) 0.5.

Therefore, if $\alpha ET_0/2(1-\nu) \le \sigma_0$, then the heat-transmission mode at R = 0 is optimal. The solution of the problem is hence easily obtained by the Fourier method [5].

Let $\alpha ET_0/2(1-\nu) > \sigma_0$. As before, we consider the optimal mode realized under the condition $\sigma_{zz}^{max} = \sigma_0$. Let us introduce the dimensionless coordinates $\xi = (x - l)/l$, $\tau = \varkappa t/l^2$. The thermoelastic stresses in a free plate due to the effect of a temperature varying only along the thickness will have the form [4]

$$\sigma_{xx} = 0,\ \sigma_{zz} = \sigma_{yy} = \frac{\alpha E}{1-\nu}\left\{-T + \frac{1}{2}\int_{-1}^{1} T d\xi + \frac{3\xi}{2}\int_{-1}^{1}\xi T d\xi\right\}.$$

Since the value of $T_1 - T_0$ is maximal at the point $\xi = -1$, then the maximal value of the thermal stress is also achieved at $\xi = -1$. Now, the problem to determine the optimal heat-transmission mode is formulated thus

$$\frac{\partial^2 U}{\partial \xi^2} = \frac{\partial U}{\partial \tau},\ U = 0 \ \text{ for } \ \tau = 0; \tag{2.2}$$

$$\frac{\partial U}{\partial \xi} = 0 \ \text{ for } \ \xi = 1;\quad -U + \frac{1}{2}\int_{-1}^{1} U d\xi - \frac{3}{2}\int_{-1}^{1}\xi U d\xi = \frac{\sigma_0(1-\nu)}{\alpha E} \ \text{ for } \ \xi = -1,$$

where $U = T_1 - T_0$.

Its solution is easily constructed by using the Laplace integral transform method

$$T_1 = T_0 + \frac{\sigma_0(1-\nu)}{\alpha E}\left\{\frac{3}{2}\left[-2\tau - (\xi-1)^2 + \frac{8}{5}\right] + 2\sum_{m=1}^{\infty}\frac{\beta_m e^{-\beta_m^2\tau}\cos\beta_m(\xi-1)}{2\beta_m\cos 2\beta_m + 2\beta_m^2\sin 2\beta_m - \sin 2\beta_m}\right\}. \tag{2.3}$$

Here β_m are positive roots of the equation

$$\beta\cos 2\beta + 3\sin^2\beta - 2\sin 2\beta = 0. \tag{2.4}$$

The values of the first 16 roots of this equation are given in the table. Subsequent roots can be calculated sufficiently accurately from the asymptotic formula

$$\beta_m \approx \pi m/2.$$

We find the control function R(t) from the condition of contact heat exchange between the plates

$$\overline{R}(\tau) = \frac{R\lambda}{l}\left(1 + \gamma\left[3\left(-\tau - \frac{6}{5}\right) + \right.\right.$$

$$\left.\left. + 2\sum_{m=1}^{\infty}\frac{\beta_m e^{-\beta_m^2\tau}\cos 2\beta_m}{2\beta_m\cos 2\beta_m + 2\beta_m^2\cos 2\beta_m - \sin 2\beta_m}\right]\right)\Big/\left(\gamma\left\{3 + \sum_{m=1}^{\infty}\frac{\beta_m^2 e^{-\beta_m^2\tau}\sin 2\beta_m}{2\beta_m\cos 2\beta_m + 2\beta_m^2\cos 2\beta_m - \sin 2\beta_m}\right\}\right), \quad (2.5)$$

where $\gamma = 2\sigma_0(1 - \nu)/\alpha E T_0$.

Since solution (2.3) is valid only for $t < t^*$, where t^* is determined from the equality

$$T_1|_{\xi=-1} = \frac{T_0}{2}, \quad (2.6)$$

then (2.5) is also valid for $t < t^*$. If the desired result is not achieved during this time, then R = 0 should later ($t > t^*$) be taken as the optimal heat-transmission mode. Let us note that the condition for switching the control (2.6) is equivalent to the equality R(t) = 0. This follows from (1.3) and the condition of heat exchange between plates.

Graphs of the control function $\overline{R}(\tau)$ are presented in the figure for different values of the parameter γ. It follows from the figure that the smaller the parameter γ, i.e., the more the thermoelastic stresses in the plate exceed the allowable magnitude under an ideal thermal contact, the longer the duration of the control.

NOTATION

T	is the temperature;
λ	is the heat-conduction coefficient;
$\varkappa$	is the coefficient of thermal diffusivity;
R	is the constant thermal resistance;
α	is the coefficient of linear expansion;
E	is the Young's modulus;
ν	is the Poisson's ratio;
$\sigma_{\gamma k}$	is the stress tensor component.

LITERATURE CITED

1. Yu. P. Shlykov and E. A. Ganin, Contact Heat Exchange [in Russian], Gosénergoizdat, Moscow–Leningrad (1963).
2. Ya. S. Podstrigach, A. G. Prokopenko, and V. M. Vigak, Prikl. Mekh., 8, No. 6 (1972).
3. V. M. Vigak, in: Mathematical Methods and Physicomechanical Fields [in Russian], No. 2, Naukova Dumka, Kiev (1975).
4. B. Boley and J. H. Weiner, Theory of Thermal Stresses, Wiley (1960).
5. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Oxford Univ. Press (1959).